SOLUTION OF A LINEAR ODE WITH INITIAL CONDITION USING LAPLACE TRANSFORMS.

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Solve the following differential equation with initial condition:

$$\begin{cases} y'' - 5y' + 6y = 2e^t \sin t \\ y(0) = 2;\ y'(0) = -3 \end{cases} . \tag{1}$$

We denote by $Y(s)$ the Laplace Transform of $y(t)$. By well-known theorems we know that:

$$\mathcal{L}\{y'(t)\}(s) = sY(s) - y(0) = sY(s) - 2$$
$$\mathcal{L}\{y''(t)\}(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s + 3$$

Moreover we know that

$$\mathcal{L}\{e^t \sin t\}(s) = \frac{1}{(s-1)^2+1}. \tag{2}$$

Applying the Laplace Transform to Eq. (**??**), we obtain:

$$\begin{aligned} s^2Y(s) - 2s + 3 - 5(sY(s) - 2) + 6Y(s) &= \frac{2}{(s-1)^2+1} \\ (s^2 - 5s + 6)Y(s) - 2s + 13 &= \frac{2}{(s-1)^2+1} \\ (s^2 - 5s + 6)Y(s) &= \frac{2}{(s-1)^2+1} + 2s - 13 \\ (s^2 - 5s + 6)Y(s) &= \frac{2s^3 - 17s^2 + 30s - 25}{(s-1)^2+1} \\ Y(s) &= \frac{2s^3 - 17s^2 + 30s - 25}{[(s-1)^2+1][s^2 - 5s + 6]}. \end{aligned}$$

We decompose $Y(s)$ into a sum of partial fractions:

$$Y(s) = \frac{3}{10}\frac{s}{(s-1)^2+1} - \frac{1}{5}\,\frac{1}{(s-1)^2+1} - \frac{34}{5}\,\frac{1}{s-3} + \frac{17}{2}\,\frac{1}{s-2}. \tag{3}$$

We know that

$$\mathcal{L}\{= e^t \sin t\}(s) = \frac{1}{(s-1)^2+1}$$
$$\mathcal{L}\{= e^t \cos t\}(s) = \frac{s-1}{(s-1)^2+1}$$

Let us write $Y(s)$ in a form more suitable for the inverse Laplace Transform:

$$\begin{aligned}
Y(s) &= \frac{3}{10}\frac{s}{(s-1)^2+1} - \frac{1}{5}\ \frac{1}{(s-1)^2+1} - \frac{34}{5}\ \frac{1}{s-3} + \frac{17}{2}\ \frac{1}{s-2} \\
&= \frac{\frac{3}{10}s - \frac{1}{5}}{(s-1)^2+1} - \frac{34}{5}\ \frac{1}{s-3} + \frac{17}{2}\ \frac{1}{s-2} \\
&= \frac{\frac{3}{10}(s-1) + \frac{1}{10}}{(s-1)^2+1} - \frac{34}{5}\ \frac{1}{s-3} + \frac{17}{2}\ \frac{1}{s-2} \\
&= \frac{3}{10}\ \frac{s-1}{(s-1)^2+1} + \frac{1}{10}\ \frac{1}{(s-1)^2+1} - \frac{34}{5}\ \frac{1}{s-3} + \frac{17}{2}\ \frac{1}{s-2}.
\end{aligned}$$

Applying the inverse Laplace Transform to each term in this sum, we obtain the solution of the differential equation with initial condition (**??**):

$$y(t) = \frac{3}{10}\ e^t\ \cos t - \frac{1}{10} e^t\ \sin t - \frac{34}{5}\ e^{3t} + \frac{17}{2}\ e^{2t}. \tag{4}$$