A NON HOMOGENEOUS LINEAR ODE WITH CONSTANT COEFFICIENTS.

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Solve the following differential equation using the method of variation of constants:

$$y''' + 3y'' - 9y' + 5y = 2e^x. \tag{1}$$

0.1. **The homogeneous equation.** We solve the homogeneous equation associated to Eq. (1)

$$y''' + 3y'' - 9y' + 5y = 0. \tag{2}$$

The characteristic equation of Eq. (2) is

$$r^3 + r^2 - 9r + 5 = 0 \tag{3}$$

An easy substitution shows that 1 is a solution of Eq. (3). By a division of polynomials we find:

$$\forall r \in \mathbb{R},\ r^3 + r^2 - 9r + 5 = (r-1)(r^2 + 4r - 5). \tag{4}$$

The solutions of Eq. (4) are 1 (double solution) and -5 (simple solution). Therefore a fundamental set of solutions (i.e. a basis for the solution space of the homogeneous equation (2) is:

$$\{e^x, xe^x, e^{-5x}\}. \tag{5}$$

The general solution of the homogeneous equation (2) is thus:

$$y_c = A_1 e^x + A_2 x e^x + A_3 e^{-5x},\ A_1, A_2, A_3 \in \mathbb{R}. \tag{6}$$

0.2. **Looking for a particular solution of the non-homogeneous equation.** We look for a particular solution of the non-homogeneous differential equation (1) in the following form:

$$y_p = u_1 e^x + u_2 x e^x + u_3 e^{-5x}, \tag{7}$$

where u_1, u_2, u_3 are functions to determine. By the method of variation of constants, we have to solve the following system of equations:

$$\begin{cases} u_1' e^x + u_2' x e^x + u_3' e^{-5x} = 0 \\ u_1' e^x + u_2'(xe^x + e^x) - 5u_3' e^{-5x} = 0 \\ u_1' e^x + u_2'(xe^x + 2e^x) + 25u_3' e^{-5x} = 2e^x \end{cases}. \tag{8}$$

The principal determinant of the system (8) is the Wronskian of the fundamental set of solutions of the homogeneous equation (2):

$$\begin{vmatrix} e^x & xe^x & e^{-5x} \\ e^x & (x+1)e^x & -5e^{-5x} \\ e^x & (x+2)e^x & 25e^{-5x} \end{vmatrix} = 36e^{-3x}. \tag{9}$$

Thus, the solution of System (8) is given by the following formulas:

$$u_1' = \frac{1}{36e^{-3x}} \begin{vmatrix} 0 & xe^x & e^{-5x} \\ 0 & (x+1)e^x & -5e^{-5x} \\ 2e^x & (x+2)e^x & 25e^{-5x} \end{vmatrix} = \frac{1}{36e^{-3x}}(-2e^{-3x}(6x+1)) = -\frac{1}{18}(6x+1),$$

$$u_2' = \frac{1}{36e^{-3x}} \begin{vmatrix} e^x & 0 & e^{-5x} \\ e^x & 0 & -5e^{-5x} \\ e^x & 2e^x & 25e^{-5x} \end{vmatrix} = \frac{1}{36e^{-3x}}(12e^{-3x}) = \frac{1}{3},$$

$$u_3' = \frac{1}{36e^{-3x}} \begin{vmatrix} e^x & xe^x & 0 \\ e^x & (x+1)e^x & 0 \\ e^x & (x+2)e^x & 2e^x \end{vmatrix} = \frac{2e^{3x}}{36e^{-3x}} = \frac{1}{18}e^{6x}.$$

We compute now integrals, in order to find the functions u_1, u_2, u_3:

$$\begin{aligned} u_1(x) &= -\frac{1}{18} \int (6x+1)\, dx = -\frac{1}{6}x^2 - \frac{1}{18}x \\ u_2(x) &= \frac{1}{3} \int dx = \frac{1}{3}x \\ u_3(x) &= \frac{1}{18} \int e^{6x}\, dx = \frac{1}{108}\, e^{6x} \end{aligned}$$

Hence, a particular solution for the non-homegeneous equation (1) is:

$$y_p = \left(-\frac{1}{6}x^2 - \frac{1}{18}x\right) e^x + \frac{1}{3}x\, xe^x + \frac{1}{108}\, e^{6x}\, e^{-5x} \tag{10}$$

$$= \left(\frac{1}{6}x^2 - \frac{1}{18}x + \frac{1}{108}\right) e^x. \tag{11}$$

Finally, we combine the results in Eq. (10) and Eq. (6); the general solution of the given non-homogeneous equation (1) is:

$$y = \left(\frac{1}{6}x^2 - \frac{1}{18}x + \frac{1}{108}\right) e^x + A_1e^x + A_2xe^x + A_3e^{-5x},\ A_1, A_2, A_3 \in \mathbb{R}. \tag{12}$$