A NON HOMOGENEOUS LINEAR ODE WITH CONSTANT COEFFICIENTS.

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Solve the following differential equation using the annihilator method:

$$y^{(4)} + 2y'' - 8y' + 5y = e^x \ \sin x. \tag{1}$$

0.1. **The homogeneous equation.** We solve the homogeneous equation associated to Eq. (**??**)

$$y^{(4)} + 2y'' - 8y' + 5y = 0. \tag{2}$$

The characteristic equation of Eq. (**??**) is

$$r^4 + 2r^2 - 8r + 5 = 0 \tag{3}$$

An easy substitution shows that 1 is a solution of Eq. (**??**). By a division of polynomials we find:

$$\forall r \in \mathbb{R},\ r^4 + 2r^2 - 8r + 5 = (r-1)(r^3 + r^2 + 3r - 5). \tag{4}$$

The second factor has 1 as a root also; once again we divide out and obtain

$$\forall r \in \mathbb{R},\ r^4 + 2r^2 - 8r + 5 = (r-1)^2(r^2 + 2r + 5). \tag{5}$$

Let us now find the roots of the second degree factor. The discriminant is given by:

$$\Delta = 2^2 - 4 \cdot 1 \cdot 5 = -16$$

Thus, the roots of the second degree factor in Eq. (**??**) are the complex numbers:

$$r = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

A fundamental set of solutions (i.e. a basis for the solution space of the homogeneous equation (**??**) is:

$$\{e^x,\ xe^x,\ e^{-x}\cos 2x,\ e^{-x}\sin 2x\}. \tag{6}$$

i.e. the general solution of the homogeneous equation (**??**) is:

$$y_c = A_1e^x + A_2xe^x + A_3e^{-x}\cos 2x + A_4e^{-x}\sin 2x,\ A_1, A_2, A_3, A_4 \in \mathbb{R}. \tag{7}$$

0.2. **Looking for a particular solution of the non-homogeneous equation.** We look for a differential operation annihilating the right-hand side of Eq. (**??**). We have:

$$\begin{aligned} u &= e^x \ \sin x \\ Du &= e^x \ \sin x + e^x \ \cos x \\ D^2u &= 2e^x \ \cos x \end{aligned}$$

The three functions u, Du, D^2u belong to the same vector space spanned by $e^x \sin x$ and $e^x \cos x$, therefore they are linearly independent. Their dependence is expressed by the following relation:

$$Du = u + \frac{1}{2}D^2u$$

which can be rewritten a follows:

$$D^2u - 2Du + 2u = 0,$$

i.e.

$$(D^2 - 2D + 2)u = 0. \tag{8}$$

The characteristic equation of Eq. (**??**) is

$$r^2 - 2r + 2 = 0$$

and the solutions of this characteristic equation are $1 \pm i$. Therefore a particular solution for Eq. (**??**) has the form

$$y_p = ae^x \sin x + be^x \cos x, \tag{9}$$

where the real coefficients a and b have to be determined. We differentiate four times y_p

$$\begin{aligned} y_p' &= (a-b)e^x \sin x + (a+b)e^x \cos x \\ y_p'' &= -2be^x \sin x + 2ae^x \cos x \\ y_p''' &= 2(a+b)e^x \sin x + 2(a-b)e^x \cos x \\ y_p^{(4)} &= 4ae^x \sin x - 4be^x \cos x \end{aligned}$$

and sustitute into Eq. (**??**):

$$\begin{gathered} (4ae^x \sin x - 4be^x \cos x) + 2(-2be^x \sin x + 2ae^x \cos x) \\ -8((a-b)e^x \sin x + (a+b)e^x \cos x) + 5(ae^x \sin x + be^x \cos x) = e^x \sin x \end{gathered}$$

i.e.

$$(7a - 4b)e^x \sin x - (4a + 7b)e^x \cos x = e^x \sin x. \tag{10}$$

As this equation must hold for every real x, we identify the corresponding coefficients in both sides and solve the system of equations obtained:

$$\begin{cases} 7a - 4b = 1 \\ 4a + 7b = 0 \end{cases}. \tag{11}$$

The solution is

$$(a, b) = \left(\frac{7}{65}, -\frac{4}{65}\right),$$

i.e.

$$y_p = \frac{7}{65}e^x \sin x - \frac{4}{65}e^x \cos x. \tag{12}$$

Combining the results of Eq. (**??**) and Eq. (**??**), we obtain the general solution of the given equation (**??**):

(13)

$$y = \frac{7}{65}e^x \sin x - \frac{4}{65}e^x \cos x + A_1e^x + A_2xe^x + A_3e^{-x}\cos 2x + A_4e^{-x}\sin 2x,\ A_1, A_2, A_3, A_4 \in \mathbb{R}.$$