The discrete continuous interplay. Will the last straw break the camel's back? *

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Abstract

We investigate the influence of the technology and in particular the influence of the discrete continuous interplay, which can be demonstrated by the technology, in enhancing students' mathematical thinking. We analyze how students' awareness of the limitation of discrete numerical methods, combined with error analysis, lead to a better understanding of the continuous methods. We identify the new potential offered by the instrumented work, the way students are influenced by their interaction with the Computer Algebra System and the presence of mental images created by this interaction, even when the computer is turned off. We also identify the inability of some students to differentiate between error due to mathematical meanings and error due to meanings specific to the "instrument". Our intention is to employ the possibilities offered by the technology, to elaborate activities based on the discrete-continuous interplay and to investigate their influence on students' thinking processes in relation to the notion of limit in the derivative concept.

Themes: 2(questions 2, 1), 1, 4 Approaches to the themes: 3, 5, 1

Introduction

The cognitive difficulties that accompany the learning of concepts that relate to the continuous such as limit and derivative are well known. Our empirical approach leads us to consider the interplay between the continuous and the discrete, and to examine how to use it to help students enhance their conceptual understanding of these central notions. The discrete continuous interplay is not new. It existed before the computer age. The founders of the mathematical theory developed numerical discrete approaches to better understand dynamic continuous processes. In the last decade, the use of technology, especially the Computer Algebra Systems (CAS), offers a new mean in the effort to overcome some of the conceptual difficulties. We focus on the "instrumentation process" (i.e. how the tool becomes an effective instrument of

* This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 1340/05).

mathematical thinking for the learner) in analyzing students' reactions in the context of activities based on the complementary aspect of discrete and continuous approaches.

The instrumentation theory

The instrumental approach is a specific approach built upon the instrumentation theory developed by Verillon and Rabardel (1995) in cognitive ergonomics and the anthropological theory developed by Chevallard (1992). The term 'instrumentation' is explained in Artigue (2002): The "instrument" is differentiated from the object, material or symbolic, on which it is based and for which the term "artifact" is used. An instrument is a mixed entity, in part an artifact, and in part cognitive schemes which make it an instrument. For a given individual, the artifact becomes an instrument through a process, called instrumental genesis. This process leads to the development or appropriation of schemes of instrumented action that progressively take shape as techniques that permit an effective response to given tasks.

The instrumentation theory focuses on the mathematical needs for instrumentation, on the status of instrumented techniques as well as on the unexpected complexity of instrumental genesis (Artigue (2002), Guin and Trouche (1999), Lagrange (2000)). We can take advantage of the new potentials offered by the instrumented work, for example, by means of discretization processes. Artigue (2002) warns us that the learner needs more specific knowledge about the way the artifact implements these discretization processes. Thus, it is important to be aware of the complexity of the instrumentation process. Working with a CAS introduces the learner to a system of "double reference" (Lagrange, 2000): on the one hand, he is introduced to mathematical meanings; on the other hand, he is introduced to meanings that are specific to the constraints of the instrument. Being aware of the limitation of the instrument might be helpful. Steiner and Dana-Picard (2004) demonstrate how to make advantage of the analysis of the error due to the limitation of the CAS to help students understand the theory of integration.

As a background to the present study, we present some cognitive difficulties that accompany the understanding of the limit concept.

Conceptualization of the continuous

In previous studies concerning the way students conceived real numbers, Kidron & Vinner (1983) observed that the infinite decimal is conceived as one of its finite approximation " *three digits after the decimal point are sufficient, otherwise it is not practical"* or as a dynamic creature which is in an unending process- a potentially infinite process: in each next stage we improve the precision with one more digit after the decimal point. This is not in accord with the mathematical view as expressed by Courant

(1937). Courant wrote that if the concept of limit yielded nothing more than the recognition that certain known numbers can be approximated to as closely as we like by certain sequences of other known numbers, we should have gained very little from it. The fruitfulness of the concept of limit in analysis rests essentially on the fact that limits of sequences of known numbers provide a means of dealing with other numbers which are not directly known or expressible. Thus, the limit concept should lead to a new entity and not just to one more digit after the decimal point.

The derivative function is defined as $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. By means of animation and elementary programming in a Computer Algebra System, students can visualize the process of $\frac{f(x+h)-f(x)}{h}$ approaching $f'(x)$ for decreasing *h*. The dynamic picture might reinforce the misconception that one can replace the limit by $\Delta y/\Delta x$ for Δx very small. How small? If we choose Δx =0.016 instead of 0.017, what will be the difference? There is a belief that *gradual causes have gradual effects* and that *small changes in a cause should produce small changes in its effect* (Stewart, 2001). This belief might explain the misconception that a change of, say, 0.001 in Δx will not produce a big change in its effect.

The discrete continuous interplay

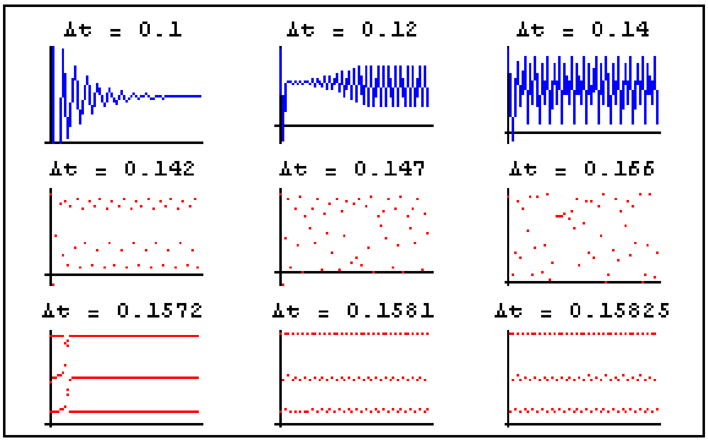
We were interested in a counterexample that will demonstrate that one cannot replace the limit "$\lim_{\Delta x\to 0}\Delta y/\Delta x$" by $\Delta y/\Delta x$ for Δx very small and that omitting the limit will change significantly the nature of the concept. The counterexample was found in the field of dynamical systems. A dynamical system is any process that evolves in time. The mathematical model is a differential equation dy/dt = y' = f(t,y) and we encounter again the derivative $y'=\lim_{\Delta t\to 0}\Delta y/\Delta t$. In a dynamical process that changes with time, time is a continuous variable. Using a numerical method to solve the differential equation, there is a discretization of the variable time and the passage to a discrete time model might totally change the nature of the solution. In the following counterexample (the logistic equation), the analytical solution obtained by means of continuous calculus is totally different from the numerical solution obtained by means of discrete numerical methods. Moreover, using the analytical solution, the students use the concept of the derivative $\lim_{\Delta x\to 0}\Delta y/\Delta x$. Using the discrete approximation by means of the numerical method the students use $\Delta y/\Delta x$ for small Δx. We will see that the two solutions, the analytical and the numerical, are totally different.

The design of the learning experiment

The learning experience is described in a pilot study (Kidron , 2003). We report it here for the convenience of the reader. The students (first year College students in a differential equations' course) were given the following task: a point (t_0, y_0) and the derivative of the function dy/dt=f(t,y) are given. Plot the function y(t). The students were asked to find the next point (t_1, y_1) by means of $(y_1 - y_0)/(t_1 - t_0) = f(t_0, y_0)$. As t increases by the small constant step $t_1 - t_0 = \Delta t$, the students realized that they are moving along the tangent line in the direction of the slope $f(t_0, y_0)$. The students generalized and wrote the algorithm: $\boldsymbol{y}_{n+1} = y_n + \Delta t\, f(t_n, \boldsymbol{y}_n)$ for Euler's method. They were asked how to better approach the solution. They proposed to choose a smaller step Δt.

The **logistic equation** dy/dt = r y(t) (1-y(t)), y(0) = y_0 was introduced as a model for the dynamics of the growth of a population. An analytical solution exists for all values of the parameter r. The numerical solution is totally different for different values of Δt as we can see in the graphical representations of the Euler's numerical solution of the logistic equation with r = 18 and y(0)=1.3.

In the first plot, the solution tends to 1 and looks like the analytical solution. In the second, third and fourth plot, the process becomes a periodic oscillation between two, four and eight levels. In the fourth plot, we did not join the points, in order that this period doubling will be clearer. In the fifth and sixth plot, the logistic mapping becomes chaotic. We slightly decrease Δt in the seventh plot. For the first 40 iterations, the logistic map appears chaotic. Then, period 3 appears. As we increase Δt very gradually we get, in the eight plot, period 6 and, in the ninth plot period 12 and **the belief that gradual causes have gradual effects is false!** The fact that a small change in a parameter causes only a small effect, does not necessarily imply that a further small change in the parameter will cause only a further small change in the effect. We knew this long ago. We just did not realize there was a mathematical consequence. Take, for example, the proverb: *the last straw breaks the camel back* (Stewart, 2001)



Findings and Discussion

First year college students in an innovative differential equations' course (N=60), were the participants in the research. The first author taught the course. The students interact with the Mathematica software in the exercise lessons which were held in the PC laboratories. Mathematica was also used during the lectures for demonstrations.
We examined the students' reactions when they realized that the approximate solution to the logistic equation by means of discrete numerical methods is so different from the analytical solution. The students were given written questionnaires before and after being exposed to the logistic equation. Some of the students were also interviewed and invited to explain their answers. We present the analysis of the students' reactions in the light of the instrumentation theory.

Awareness of the limitations of the numerical method

Before being introduced to the logistic equation, the students were asked if a very small value for the step size Δt in Euler's method will assure a good approximation to the solution: 80% of the students claimed that a small value for Δt might be not small enough. They connect their claim to the limitations of the numerical method. Some students also pointed out the fact that the error in the numerical method accumulates.

The influence of previous experiences in the PC lab on the students thinking processes

The students were influenced by previous experiences in the PC lab even if these experiences took place in other courses.

> Irit: We worked on several examples in which we noticed that the more points, the smaller Δt and the better the approximation BUT I think that not every function will behave this way. We encountered in the lab, in relation to another subject, a function with a special behavior: In spite of the fact that we added interpolation points, the function did not behave the way we expected. I think it was $f(x) = \frac{1}{(1+x^2)}$. Usually, the more points, the better is the approximation but it is not always the case.

Irit referred to Runge's example. The students worked in the lab looking for polynomial approximations to the function with equidistant interpolation points.

The belief that gradual causes have gradual effects

Before being exposed to the logistic equation, the students were asked to express their opinion about the following statement:" If in Euler's method, using a step size $\Delta t = 0.017$ we get a solution very far from the real solution, then a step size $\Delta t = 0.016$ will not produce a big improvement, maybe some digits after the decimal point and no more". The belief that gradual causes have gradual effects was expressed in 53%

of the students' answers "*it seems to me that if with* $\Delta t = 0.017$ *we didn't get a good solution, then* $\Delta t = 0.016$ *will not produce a big improvement either"*. 31% of the students who claimed that gradual causes have no necessarily gradual effects explained their answer by the fact that "*There are functions which oscillate very quickly"* or by means of the accumulating effect

> "I think that we use an iterative procedure to find y_{k+1}, namely, we perform the algorithm on y_k in order to find y_{k+1} and so on… After several iterations, we accumulate differences in the value of y_{k+1} that might be significant therefore we might get a big improvement even with a slightly smaller Δt".

The (in)ability to differentiate between error due to mathematical meaning and error due to the instrument

After being exposed to the logistic equation, the students were asked to characterize the source of error in Euler's method. We investigate whether the students realize that the source of error is the fact that in the numerical method the limit has been omitted in the definition of the derivative.

- **The students' attention might be distracted by the round off error**

The students' attention might be distracted by the round-off error especially if in previous experience with the computer they encountered such kind of round off error. This happened to a student, Hadas, which attributed the error to the round-off effect

> Hadas: I remember from an exercise in the Calculus course that the solution with Matlab was 0 but the solution using the symbolic form was 0.5. When we tried to understand why this happened we realized that MatLab computes only 15 digits after the decimal point.

Hadas refered to an episode in the Calculus course in which the students were given the function $f(x) = (1 - \cos(x^6))/x^{12}$ and they had to explain why some graphs of f might give false information about $\lim_{x \to 0} f(x)$. The limit is ½ but both Mathematica and MatLab give the answer 0 when we evaluate the function for x = 0.01. Working the exercise in the PC lab, the students understood that the computer with its limited precision gives the incorrect result that $1 - \cos(x^6)$ is 0 for even moderately small values of x.

By means of error analysis, we planned to help the students to better understand the continuous methods and the concept of limit. But, working with a CAS, there are other unexpected effects that are directly linked with the "instrument" and the way it influences the students' thinking. In addition to the error due to the discretization process, to the fact that an algorithm that belongs to a numerical method is used to solve the logistic equation in place of the analytical method, there are other sources of error that are directly related to the "tool".

- **The students' attention might be distracted by the accumulative effect**

22% of the students explained the error by means of the accumulative effect.

- **Round off and accumulating effect could not be the only source of error but the students cannot find the exact source of error**

This view was expressed in answers like the following:

"The software has some limitation concerning the number of digits after the decimal point and it seems that this fact has enough strength (due to the accumulative effect) to influence the solution with Euler's method. But this fact by itself could not cause the crazy behavior of the function" or:

> "The round-off is not the crucial part. There is a change in the analytical behavior. We do expect for a change due to the round off, but we expect to a change "in numbers" not in the qualitative behavior".

- **The students relate the error to discrete - continuous considerations, but without a mention of the limit or of the formal definition of the derivative.**

23% of the answers expressed well developed qualitative approach to differential equations, adequate to explain why there is an error, but inadequate to give a formal account how the discrete method employed the derivative concept.

- **The students relate the error to the fact that in the numerical solution the limit is omitted in the definition of the derivative.**

This was expressed in 19% of the answers. Some students reached this conclusion by means of mental images created by previous experiences with the instrument. These images were also present when the computer was turned off. This is expressed in the following:

> "We have worked this week an exercise that demonstrates that a small change in the initial condition of a differential equation might cause a large change in the solution. Maybe the small error made in the Euler's method induced big changes in the graph of the solution curve also in our case" or:
>
> "A difference of 0.001 might be crucial if it leads to the crossing of an equilibrium solution and therefore to a transition to a zone with totally different slopes like in the example.."

The student related to a figure describing a previous experience in the lab

> "In the absence of an equilibrium solution, the error would have increased but there would not have been crucial changes. The error increased but we noticed it only because the equilibrium solution". In connection to the metaphor of the straw and the camel's back, the student added: *if there was no equilibrium line, the camel would not have fall down!*

Concluding remarks: The emphasis in this research study is laid on the way we take advantage of the discrete continuous interplay *to identify* *the new potentials offered by instrumented work* but also on *identifying the constraints induced by the instrument*. The “instrument” plays a very important role. It enables the students to “see” the significant difference between the discrete and the continuous methods. It also helps the students to analyze qualitatively the behavior of the solution and to connect it with former experiences. Mathematica enables to change very slightly the value of a parameter and to plot the solution. The role played by the instrument is very important and enable the new potentials offered by the instrumented work. However, the percentage of students who did not find the source of error in the numerical method, demonstrated that it might have not been enough to expose the students to a counterexample. Students have to be faced with the necessity of developing schemes that will help them to differentiate between error due to mathematical meanings and error due to meanings specific to the instrument (Artigue, 2002).

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